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Noncommutative Spaces and Gravity

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Abstract

We give an introduction to an algebraic construction of a gravity theory on noncommutative spaces which is based on a deformed algebra of (infinitesimal) diffeomorphisms. We start with some fundamental ideas and concepts of noncommutative spaces. Then the θ-deformation of diffeomorphisms is studied and a tensor calculus is defined. A deformed Einstein-Hilbert action invariant with respect to deformed diffeomorphisms is given. Finally, all noncommutative fields are expressed in terms of their commutative counterparts up to second order of the deformation parameter using the $\star$-product. This allows to study explicitly deviations to Einstein's gravity theory in orders of θ. This lecture is based on joined work with P. Aschieri, C. Blohmann, M. Dimitrijević, P. Schupp and J. Wess.

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Contents

1 Noncommutative Spaces

In field theories one usually considers differential space-time manifolds. In the noncommutative realm, the notion of a point is no longer well-defined and we have to give up the concept of differentiable manifolds. However, the space of functions on a manifold is an algebra. A generalization of this algebra can be considered in the noncommutative case. We take the algebra freely generated by the *noncommutative coordinates* $\hat{x}^i$ which respects commutation relations of the type

$$[\hat{x}^\mu, \hat{x}^\nu] = C^{\mu\nu}(\hat{x}) \neq 0\,. \tag{1}$$

Without bothering about convergence, we take the space of formal power series in the coordinates $\hat{x}^i$ and divide by the ideal generated by the above relations

$$\widehat{\mathcal{A}}_{\hat{x}} = \mathbb{C}\langle\langle \hat{x}^0, \ldots, \hat{x}^n \rangle\rangle / ([\hat{x}^\mu, \hat{x}^\nu] - C^{\mu\nu}(\hat{x}))\,.$$

The function $C^{\mu\nu}(\hat{x})$ is unknown. For physical reasons it should be a function that vanishes at large distances where we experience the commutative world and may be determined by experiments. Nevertheless, one can consider a power-series expansion

$$C^{\mu\nu}(\hat{x}) = i\theta^{\mu\nu} + iC^{\mu\nu}{}_\rho \hat{x}^\rho + (q\hat{R}^{\mu\nu}{}_{\rho\sigma} - \delta^\nu_\rho \delta^\mu_\sigma)\hat{x}^\rho \hat{x}^\sigma + \ldots,$$

where $\theta^{\mu\nu}$, $C^{\mu\nu}{}_\rho$ and $q\hat{R}^{\mu\nu}{}_{\rho\sigma}$ are constants, and study cases where the commutation relations are constant, linear or quadratic in the coordinates. At very short distances those cases provide a reasonable approximation for $C^{\mu\nu}(\hat{x})$ and lead to the following three structures

1. canonical or θ-deformed case:
$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \tag{2}$$

2. Lie algebra case:
$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}{}_\rho \hat{x}^\rho. \tag{3}$$

3. Quantum Spaces:
$$\hat{x}^\mu \hat{x}^\nu = q\hat{R}^{\mu\nu}{}_{\rho\sigma}\hat{x}^\rho \hat{x}^\sigma. \tag{4}$$

We denote the algebra generated by noncommutative coordinates $\hat{x}^\mu$ which are subject to the relations (2) by $\hat{\mathcal{A}}$. We shall often call it the *algebra of noncommutative functions*. Commutative functions will be denoted by $\mathcal{A}$. In what follows we will exclusively consider the θ-deformed case (2) but we note that the algebraic construction presented here can be generalized to more complicated noncommutative structures of the above type which possess the Poincaré-Birkhoff-Witt (PBW) property. The PBW-property states that the space of polynomials in noncommutative coordinates of a given degree is isomorphic to the space of polynomials in the commutative coordinates. Such an isomorphism between polynomials of a fixed degree is given by an ordering prescription. One example is the *symmetric ordering* (or Weyl-ordering) W which assigns to any monomial the totally symmetric ordered monomial

$$\begin{array}{rcl} W : \mathcal{A} & \to & \hat{\mathcal{A}} \\ x^\mu & \mapsto & \hat{x}^\mu \\ x^\mu x^\nu & \mapsto & \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + \hat{x}^\nu \hat{x}^\mu) \\ & \cdots & . \end{array} \tag{5}$$

To study the dynamics of fields we need a differential calculus on the noncommutative algebra $\hat{\mathcal{A}}$. Derivatives are maps on the deformed coordinate space [1]

$$\hat{\partial}_\mu : \hat{\mathcal{A}} \to \hat{\mathcal{A}}\,.$$

This means that they have to be consistent with the commutation relations of the coordinates. In the θ-constant case a consistent differential calculus can be defined very easily by[1]

$$\begin{array}{l} [\hat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu \qquad (\hat{\partial}_\mu \hat{x}^\nu) = \delta_\mu^\nu \\ [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0. \end{array} \tag{6}$$

[1]We use brackets to distinguish the action of a differential operator from the multiplication in the algebra of differential operators.

It is the fully undeformed differential calculus. The above definitions yield the usual Leibniz-rule for the derivatives $\hat{\partial}_\mu$

$$(\hat{\partial}_\mu \hat{f}\hat{g}) = (\hat{\partial}_\mu \hat{f})\hat{g} + \hat{f}(\hat{\partial}_\mu \hat{g}). \tag{7}$$

This is a special feature of the fact that $\theta^{\mu\nu}$ are constants. In the more complicated examples of noncommutative structures this undeformed Leibniz-rule usually cannot be preserved but one has to consider deformed Leibniz-rules for the derivatives [2]. Note that (6) also implies that

$$(\hat{\partial}_\mu \hat{f}) = \widehat{(\partial_\mu f)}. \tag{8}$$

The Weyl ordering (5) can be formally implemented by the map

$$f \mapsto W(f) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int d^n k \, e^{ik_\mu \hat{x}^\mu} \tilde{f}(k)$$

where $\tilde{f}$ is the Fourier transform of f

$$\tilde{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int d^n x \, e^{-ik_\mu x^\mu} f(x).$$

This is due to the fact that the exponential is a fully symmetric function. Using the Baker-Campbell-Hausdorff formula one finds

$$e^{ik_\mu \hat{x}^\mu} e^{ip_\nu \hat{x}^\nu} = e^{i(k_\mu + p_\mu)\hat{x}^\mu - \frac{i}{2} k_\mu \theta^{\mu\nu} p_\nu}. \tag{9}$$

This immediately leads to the following observation

$$\begin{aligned} \hat{f}\hat{g} &= W(f)W(g) = \frac{1}{(2\pi)^n} \int d^n k d^n p \, e^{ik_\mu \hat{x}^\mu} e^{ip_\nu \hat{x}^\nu} \tilde{f}(k)\tilde{g}(p) \\ &= \frac{1}{(2\pi)^n} \int d^n k d^n p \, e^{i(k_\mu + p_\mu)\hat{x}^\mu} e^{-\frac{i}{2} k_\mu \theta^{\mu\nu} p_\nu} \tilde{f}(k)\tilde{g}(p) \\ &= W(\mu \circ e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g), \end{aligned} \tag{10}$$

where $\mu(f \otimes g) := fg$ is the multiplication map. With (8) we deduce from (10) the equation

$$\mu \circ e^{-\frac{i}{2}\theta^{\mu\nu}\hat{\partial}_\mu \otimes \hat{\partial}_\nu} \hat{f} \otimes \hat{g} = \widehat{fg}. \tag{11}$$

The above formula shows us how the commutative and the noncommutative product are related. It will be important for us later on.

2 Symmetries on Deformed Spaces

In general the commutation relations (1) are not covariant with respect to undeformed symmetries. For example the canonical commutation relations (2) break Lorentz symmetry if we assume that the noncommutativity parameters $\theta^{\mu\nu}$ do not transform.

The question arises whether we can *deform* the symmetry in such a way that it acts consistently on the deformed space (i.e. leaves the deformed space invariant) and such that it reduces to the undeformed symmetry in the commutative limit. The answer is yes: Lie algebras can be deformed in the category of Hopf algebras (Hopf algebras coming from a Lie algebra are also called Quantum Groups)[2]. Important examples of such deformations are q-deformations: Drinfeld and Jimbo have shown that there exists a q-deformation of the universal enveloping algebra of an arbitrary semisimple Lie algebra[3]. Module algebras of this q-deformed universal enveloping algebras are noncommutative spaces with commutation relations of type (4). There exists also a so-called κ-deformation of the Poincaré algebra [3, 4] which leads to a noncommutative space of the Lie type (3). A Hopf algebra symmetry acting on the θ-deformed space was for a long time unknown. But recently also a θ-deformation of the Poincaré algebra leading to the algebra (2) was constructed [5–8].

Quantum group symmetries lead to new features of field theories on noncommutative spaces. Because of its simplicity, θ-deformed spaces are very well-suited to study those. First results on the consequences of the θ-deformed Poincaré algebra have already been obtained [6, 8]. However, it remains unknown and subject of future investigations in which precise way this recently discovered quantum group symmetry restricts the degrees of freedom of the noncommutative field theory.

In the following we will construct explicitly a θ-deformed version of diffeomorphisms which consistently act on the noncommutative space (2). Then we present a gravity theory which is invariant with respect to this deformed diffeomorphisms [8,9].

3 Diffeomorphisms

Gravity is a theory invariant with respect to diffeomorphisms. However, to generalize the Einstein formalism to noncommutative spaces in order to establish a gravity

[2]To be more precise the universal enveloping algebra of a Lie algebra can be deformed. The universal enveloping algebra of any Lie algebra is a Hopf algebra and this gives rise to deformations in the category of Hopf algebras.

[3]It is called q-deformation since it is a deformation in terms of a parameter q.

theory, it is important to first understand that diffeomorphisms possess more mathematical structure than the algebraic one: They are naturally equipped with a Hopf algebra structure. In the common formulations of physical theories this additional Hopf structure is hidden and does not play a crucial role. It is our aim to deform the algebra of diffeomorphism in such a way that it acts consistently on a noncommutative space. This can be done by exploiting the full Hopf structure. In this section we first introduce the concept of diffeomorphisms as Hopf algebra in the undeformed setting.

Diffeomorphisms are generated by vector-fields ξ. Acting on functions, vector-fields are represented as linear differential operators $\xi = \xi^\mu \partial_\mu$. Vector-fields form a Lie algebra Ξ over the field $\mathbb{C}$ with the Lie bracket given by

$$[\xi, \eta] = \xi \times \eta$$

where $\xi \times \eta$ is defined by its action on functions

$$(\xi \times \eta)(f) = (\xi^\mu (\partial_\mu \eta^\nu)\partial_\nu - \eta^\mu (\partial_\mu \xi^\nu)\partial_\nu)(f).$$

The Lie algebra of *infinitesimal diffeomorphisms* Ξ can be embedded into its universal enveloping algebra which we want to denote by $\mathcal{U}(\Xi)$. The universal enveloping algebra is an associative algebra and possesses a natural Hopf algebra structure. It is given by the following structure maps[4]:

- An algebra homomorphism called *coproduct* defined by

$$\begin{aligned} \Delta : \mathcal{U}(\Xi) &\rightarrow \mathcal{U}(\Xi) \otimes \mathcal{U}(\Xi) \\ \Xi \ni \xi &\mapsto \Delta(\xi) := \xi \otimes 1 + 1 \otimes \xi. \end{aligned} \tag{12}$$

- An algebra homomorphism called *counit* defined by

$$\begin{aligned} \varepsilon : \mathcal{U}(\Xi) &\rightarrow \mathbb{C} \\ \Xi \ni \xi &\mapsto \varepsilon(\xi) = 0. \end{aligned} \tag{13}$$

- An anti-algebra homomorphism called *antipode* defined by

$$\begin{aligned} S : \mathcal{U}(\Xi) &\rightarrow \mathcal{U}(\Xi) \\ \Xi \ni \xi &\mapsto S(\xi) = -\xi. \end{aligned} \tag{14}$$

[4]The structure maps are defined on the generators $\xi \in \Xi$ and the universal property of the universal enveloping algebra $\mathcal{U}(\Xi)$ assures that they can be uniquely extended as algebra homomorphisms (respectively anti-algebra homomorphism in case of the antipode S) to the whole algebra $\mathcal{U}(\Xi)$.

For a precise definition and more details on Hopf algebras we refer the reader to text books [10–12]. For our purposes it shall be sufficient to note that the coproduct implements how the Hopf algebra acts on a product in a representation algebra (Leibniz-rule). Below we will make this more transparent. It is now possible to study deformations of $\mathcal{U}(\Xi)$ in the category of Hopf algebras. This leads to a deformed version of diffeomorphisms - the fundamental building block of our approach to a gravity theory on noncommutative spaces. Before studying this in detail, let us shortly review the Einstein formalism. This way we first understand better the meaning of the structure maps of a Hopf algebra introduced above.

Scalar fields are defined by their transformation property with respect to infinitesimal coordinate transformations:

$$\delta_\xi \phi = -\xi\phi = -\xi^\mu(\partial_\mu \phi). \tag{15}$$

The product of two scalar fields is transformed using the Leibniz-rule

$$\delta_\xi(\phi\psi) = (\delta_\xi\phi)\psi + \phi(\delta_\xi\psi) = -\xi^\mu(\partial_\mu\phi\psi) \tag{16}$$

such that the product of two scalar fields transforms again as a scalar. The above Leibniz-rule can be understood in mathematical terms as follows: The Hopf algebra $\mathcal{U}(\Xi)$ is represented on the space of scalar fields by infinitesimal coordinate transformations δ_ξ. On scalar fields the action of δ_ξ is explicitly given by the differential operator $-\xi^\mu\partial_\mu$. Of course, the space of scalar fields is not only a vector space - it possesses also an algebra structure - such as $\mathcal{U}(\Xi)$ is not only an algebra but also a Hopf algebra - it possesses in addition the co-structure maps defined above. We say that a Hopf algebra H acts on an algebra A (or more precisely we say that A is a left H-module algebra) if A is a module with respect to the algebra H and if in addition for all $h \in H$ and $a, b \in A$

$$\begin{aligned} h(ab) &= \mu \circ \Delta h(a \otimes b) && (17)\\ h(1) &= \varepsilon(h). && (18)\end{aligned}$$

Here μ is the multiplication map defined by $\mu(a \otimes b) = ab$. In our concrete example where $H = \mathcal{U}(\Xi)$ and A is the algebra of scalar fields we indeed have that the algebra of scalar fields is a $\mathcal{U}(\Xi)-$module algebra. This can be seen easily if we rewrite (16) using (12) for the generators $\xi \in \Xi$ for $\mathcal{U}(\Xi)$:

$$\delta_\xi(\phi\psi) = (\delta_\xi\phi)\psi + \phi(\delta_\xi\psi) = \mu \circ \Delta\xi(\phi \otimes \psi).$$

It is also evident that

$$\delta_\xi 1 = 0 = \varepsilon(\xi)1.$$

Now we are in the right mathematical framework: We study a Lie algebra (here infinitesimal diffeomorphisms Ξ) and embed it in its universal enveloping algebra (here $\mathcal{U}(\Xi)$). This universal enveloping algebra is a Hopf algebra via a natural Hopf structure induced by (12,13,14).

Physical quantities live in representations of this Hopf algebras. For instance, the algebra of scalar fields is a $\mathcal{U}(\Xi)$-module algebra. The action of $\mathcal{U}(\Xi)$ on scalar fields is given in terms of infinitesimal coordinate transformations δ_ξ.

Similarly one studies tensor representations of $\mathcal{U}(\Xi)$. For example vector fields are introduced by the transformation property

$$\begin{aligned} \delta_\xi V_\alpha &= -\xi^\mu(\partial_\mu V_\alpha) - (\partial_\alpha \xi^\mu) V_\mu \\ \delta_\xi V^\alpha &= -\xi^\mu(\partial_\mu V^\alpha) + (\partial_\mu \xi^\alpha) V^\mu . \end{aligned}$$

The generalization to arbitrary tensor fields is straight forward:

$$\begin{aligned} \delta_\xi T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} &= -\xi^\mu(\partial_\mu T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n}) + (\partial_\mu \xi^{\mu_1}) T^{\mu \cdots \mu_n}_{\nu_1 \cdots \nu_n} + \cdots + (\partial_\mu \xi^{\mu_n}) T^{\mu_1 \cdots \mu}_{\nu_1 \cdots \nu_n} \\ &\quad -(\partial_{\nu_1} \xi^\nu) T^{\mu_1 \cdots \mu_n}_{\nu \cdots \nu_n} - \cdots - (\partial_{\nu_n} \xi^\nu) T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu} . \end{aligned}$$

As for scalar fields, we also find that the product of two tensors transforms like a tensor. Summarizing, we have seen that scalar fields, vector fields and tensor fields are representations of the Hopf algebra $\mathcal{U}(\Xi)$, the universal enveloping algebra of infinitesimal diffeomorphisms. The Hopf algebra $\mathcal{U}(\Xi)$ acts via *infinitesimal coordinate transformations* δ_ξ which are subject to the relations:

$$\begin{aligned} [\delta_\xi, \delta_\eta] = \delta_{\xi \times \eta} \qquad & \varepsilon(\delta_\xi) = 0 \\ \Delta \delta_\xi = \delta_\xi \otimes 1 + 1 \otimes \delta_\xi \qquad & S(\delta_\xi) = -\delta_\xi . \end{aligned} \tag{19}$$

The transformation operator δ_ξ is explicitly given by differential operators which depend on the representation under consideration. In case of scalar fields this differential operator is given by $-\xi^\mu \partial_\mu$.

4 Deformed Diffeomorphisms

The concepts introduced in the previous subsection can be deformed in order to establish a consistent tensor calculus on the noncommutative space-time algebra (2). In this context it is necessary to account the full Hopf algebra structure of the universal enveloping algebra $\mathcal{U}(\Xi)$.

In our setting the algebra $\hat{\mathcal{A}}$ possesses a noncommutative product defined by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{20}$$

We want to deform the structure maps (19) of the Hopf algebra $\mathcal{U}(\Xi)$ in such a way that the resulting deformed Hopf algebra which we denote by $\mathcal{U}(\hat{\Xi})$ consistently acts on $\hat{\mathcal{A}}$. In the language introduced in the previous section this means that we want $\hat{\mathcal{A}}$ to be a $\mathcal{U}(\hat{\Xi})$-module algebra. We claim that the following deformation of $\mathcal{U}(\Xi)$ does the job. Let $\mathcal{U}(\hat{\Xi})$ be generated as algebra by elements $\hat{\delta}_\xi$, $\xi \in \Xi$. We leave the algebra relation undeformed and demand

$$[\hat{\delta}_\xi, \hat{\delta}_\eta] = \hat{\delta}_{\xi\times\eta} \tag{21}$$

but we deform the co-sector

$$\Delta\hat{\delta}_\xi = e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}(\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi)e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho\otimes\hat{\partial}_\sigma}, \tag{22}$$

where

$$[\hat{\partial}_\rho, \hat{\delta}_\xi] = \hat{\delta}_{(\partial_\rho\xi)}.$$

The deformed coproduct (22) reduces to the undeformed one (19) in the limit $\theta \to 0$. Antipode and counit remain undeformed

$$S(\hat{\delta}_\xi) = -\hat{\delta}_\xi \qquad \varepsilon(\hat{\delta}_\xi) = 0. \tag{23}$$

We have to check whether the above deformation is a good one in the sense that it leads to a consistent action on $\hat{\mathcal{A}}$. First we need a differential operator acting on fields in $\hat{\mathcal{A}}$ which represents the algebra (21). Let us consider the differential operator

$$\hat{X}_\xi := \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{2})^n \theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}(\hat{\partial}_{\rho_1}\cdots\hat{\partial}_{\rho_n}\hat{\xi}^\mu)\hat{\partial}_\mu\hat{\partial}_{\sigma_1}\cdots\hat{\partial}_{\sigma_n}. \tag{24}$$

This is to be understood like that: A vector-field $\xi = \xi^\mu\partial_\mu$ is determined by its coefficient functions ξ^μ. In Section 1 we saw that there is a vectorspace isomorphism W from the space of commutative to the space of noncommutative functions which is given by the symmetric ordering prescription. The image of a commutative function f under the isomorphism W is denoted by $\hat{f}$

$$W:\ f \mapsto W(f) = \hat{f}.$$

In (24) $\hat{\xi}^\mu$ is therefore to be interpreted as the image of ξ^μ with respect to W. Then indeed we have

$$[\hat{X}_\xi, \hat{X}_\eta] = \hat{X}_{\xi\times\eta}. \tag{25}$$

To see this we use result (11) to rewrite $(\hat{X}_\xi \hat{\phi})$:

$$\begin{aligned}
(\hat{X}_\xi \hat{\phi}) &= \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{2})^n \theta^{\rho_1\sigma_1} \cdots \theta^{\rho_n\sigma_n} (\hat{\partial}_{\rho_1} \cdots \hat{\partial}_{\rho_n} \hat{\xi}^\mu)(\hat{\partial}_\mu \hat{\partial}_{\sigma_1} \cdots \hat{\partial}_{\sigma_n} \hat{\phi}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{2})^n \theta^{\rho_1\sigma_1} \cdots \theta^{\rho_n\sigma_n} (\hat{\partial}_{\rho_1} \cdots \hat{\partial}_{\rho_n} \hat{\xi}^\mu)(\hat{\partial}_{\sigma_1} \cdots \hat{\partial}_{\sigma_n} \widehat{\partial_\mu \phi}) \\
&= \widehat{\xi^\mu (\partial_\mu \phi)} = \widehat{(\xi \phi)}. \qquad (26)
\end{aligned}$$

From (26) follows

$$(\hat{X}_\xi (\hat{X}_\eta \hat{\phi})) - (\hat{X}_\eta (\hat{X}_\xi \hat{\phi})) = (\widehat{[\xi,\eta]\phi}) = (\hat{X}_{\xi\times\eta} \hat{\phi}),$$

which amounts to (25) and this is what we wanted to show.

It is therefore reasonable to introduce scalar fields $\hat{\phi} \in \hat{\mathcal{A}}$ by the transformation property

$$\hat{\delta}_\xi \hat{\phi} = -(\hat{X}_\xi \hat{\phi}).$$

The next step is to work out the action of the differential operators $\hat{X}_\xi$ on the product of two fields. A calculation [8] shows that

$$(\hat{X}_\xi (\hat{\phi}\hat{\psi})) = \mu \circ (e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma} (\hat{X}_\xi \otimes 1 + 1 \otimes \hat{X}_\xi) e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma} \hat{\phi} \otimes \hat{\psi}).$$

This means that the differential operators $\hat{X}_\xi$ act via a *deformed Leibniz rule* on the product of two fields. Comparing with (22) we see that the deformed Leibniz rule of the differential operator $\hat{X}_\xi$ is exactly the one induced by the deformed coproduct (22):

$$\hat{\delta}_\xi (\hat{\phi}\hat{\psi}) = e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma} (\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi) e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma} (\hat{\phi}\hat{\psi}) = -\hat{X}_\xi \triangleright (\hat{\phi}\hat{\psi}).$$

Hence, the deformed Hopf algebra $\mathcal{U}(\hat{\Xi})$ is indeed represented on scalar fields $\hat{\phi} \in \hat{\mathcal{A}}$ by the differential operator $\hat{X}_\xi$. The scalar fields form a $\mathcal{U}(\hat{\Xi})$-module algebra.

In analogy to the previous section we can introduce vector and tensor fields as representations of the Hopf algebra $\mathcal{U}(\hat{\Xi})$. The transformation property for an arbitrary tensor reads

$$\begin{aligned}
\hat{\delta}_\xi \hat{T}^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s} &= -(\hat{X}_\xi \hat{T}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}) + (\hat{X}_{(\partial_\mu \xi^{\mu_1})} \hat{T}^{\mu\cdots\mu_n}_{\nu_1\cdots\nu_n}) + \cdots + (\hat{X}_{(\partial_\mu \xi^{\mu_n})} \hat{T}^{\mu_1\cdots\mu}_{\nu_1\cdots\nu_n}) \\
&\quad -(\hat{X}_{(\partial_{\nu_1} \xi^{\nu})} \hat{T}^{\mu_1\cdots\mu_n}_{\nu\cdots\nu_n}) - \cdots - (\hat{X}_{(\partial_{\nu_n} \xi^{\nu})} \hat{T}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu}).
\end{aligned}$$

Up to now we have seen the following:

- Diffeomorphisms are generated by vector-fields $\xi \in \Xi$ and the universal enveloping algebra $\mathcal{U}(\Xi)$ of the Lie algebra Ξ of vector-fields possesses a natural Hopf algebra structure defined by (19).
- The algebra of scalar fields $\phi \in \mathcal{A}$ is a $\mathcal{U}(\Xi)$-module algebra.
- The universal enveloping algebra $\mathcal{U}(\Xi)$ can be deformed to a Hopf algebra $\mathcal{U}(\hat{\Xi})$ defined in (21,22,23).
- $\mathcal{U}(\hat{\Xi})$ consistently acts on the algebra of noncommutative functions $\hat{\mathcal{A}}$, i.e. the algebra of noncommutative functions is a $\mathcal{U}(\hat{\Xi})$-module algebra.
- Regarding $\mathcal{U}(\hat{\Xi})$ as the underlying "symmetry" of the gravity theory to be built on the noncommutative space $\hat{\mathcal{A}}$, we established a full tensor calculus as representations of the Hopf algebra $\mathcal{U}(\hat{\Xi})$.

5 Noncommutative Geometry

The deformed algebra of infinitesimal diffeomorphisms and the tensor calculus covariant with respect to it is the fundamental building-block for the definition of a noncommutative geometry on $\theta-$deformed spaces. In this section we sketch the important steps towards a deformed Einstein-Hilbert action [8]. A first ingredient is the *covariant derivative* $\hat{D}_\mu$. Algebraically, it can be defined by demanding that acting on a vector-field it produces a tensor-field

$$\hat{\delta}_\xi \hat{D}_\mu \hat{V}_\nu \stackrel{!}{=} -(\hat{X}_\xi \hat{D}_\mu \hat{V}_\nu) - (\hat{X}_{(\partial_\mu \xi^\alpha)} \hat{D}_\alpha \hat{V}_\nu) - (\hat{X}_{(\partial_\nu \xi^\alpha)} \hat{D}_\mu \hat{V}_\alpha) \tag{27}$$

The covariant derivative is given by a *connection* $\hat{\Gamma}_{\mu\nu}{}^\rho$

$$\hat{D}_\mu \hat{V}_\nu = \hat{\partial}_\mu \hat{V}_\nu - \hat{\Gamma}_{\mu\nu}{}^\rho \hat{V}_\rho.$$

From (27) it is possible to deduce the transformation property of $\hat{\Gamma}_{\mu\nu}{}^\rho$

$$\hat{\delta}_\xi \hat{\Gamma}_{\mu\nu}{}^\rho = (\hat{X}_\xi \hat{\Gamma}_{\mu\nu}{}^\rho) - (\hat{X}_{(\partial_\mu \xi^\alpha)} \hat{\Gamma}_{\alpha\nu}{}^\rho) - (\hat{X}_{(\partial_\nu \xi^\alpha)} \hat{\Gamma}_{\mu\alpha}{}^\rho) + (\hat{X}_{(\partial_\alpha \xi^\rho)} \hat{\Gamma}_{\mu\nu}{}^\alpha) - (\hat{\partial}_\mu \hat{\partial}_\nu \hat{\xi}^\rho).$$

The *metric* $\hat{G}_{\mu\nu}$ is defined as a symmetric tensor of rank two. It can be obtained for example by a set of vector-fields $\hat{E}_\mu{}^a$, $a = 0, \ldots, 3$, where a is to be understood as a mere label. These vector-fields are called *vierbeins*. Then the symmetrized product of those vector-fields is indeed a symmetric tensor of rank two

$$\hat{G}_{\mu\nu} := \frac{1}{2}(\hat{E}_\mu{}^a \hat{E}_\nu{}^b + \hat{E}_\nu{}^b \hat{E}_\mu{}^a)\eta_{ab}.$$

Here η_{ab} stands for the usual flat Minkowski space metric with signature $(-+++)$. Let us assume that we can choose the vierbeins $\hat{E}_\mu{}^a$ such that they reduce in the commutative limit to the usual vierbeins $e_\mu{}^a$. Then also the metric $\hat{G}_{\mu\nu}$ reduces to the usual, undeformed metric $g_{\mu\nu}$.

The inverse metric tensor we denote by upper indices

$$\hat{G}_{\mu\nu}\hat{G}^{\nu\rho} = \delta_\mu^\rho.$$

We use $\hat{G}_{\mu\nu}$ respectively $\hat{G}^{\mu\nu}$ to raise and lower indices.

The curvature and torsion tensors are obtained by taking the commutator of two covariant derivatives[5]

$$[\hat{D}_\mu, \hat{D}_\nu]\hat{V}_\rho = \hat{R}_{\mu\nu\rho}{}^\alpha \hat{V}_\alpha + \hat{T}_{\mu\nu}{}^\alpha \hat{D}_\alpha \hat{V}_\rho$$

which leads to the expressions

$$\begin{aligned}
\hat{R}_{\mu\nu\rho}{}^\sigma &= \hat{\partial}_\nu \hat{\Gamma}_{\mu\rho}{}^\sigma - \hat{\partial}_\mu \hat{\Gamma}_{\nu\rho}{}^\sigma + \hat{\Gamma}_{\nu\rho}{}^\beta \hat{\Gamma}_{\mu\beta}{}^\sigma - \hat{\Gamma}_{\mu\rho}{}^\beta \hat{\Gamma}_{\nu\beta}{}^\sigma \\
\hat{T}_{\mu\nu}{}^\alpha &= \hat{\Gamma}_{\nu\mu}{}^\alpha - \hat{\Gamma}_{\nu\mu}{}^\alpha.
\end{aligned}$$

If we assume the *torsion-free* case, i.e.

$$\hat{\Gamma}_{\mu\nu}{}^\sigma = \hat{\Gamma}_{\nu\mu}{}^\sigma,$$

we find an unique expression for the metric connection (Christoffel symbol) defined by

$$\hat{D}_\alpha \hat{G}_{\beta\gamma} \stackrel{!}{=} 0$$

in terms of the metric and its inverse[6]

$$\hat{\Gamma}_{\alpha\beta}{}^\sigma = \frac{1}{2}(\hat{\partial}_\alpha \hat{G}_{\beta\gamma} + \hat{\partial}_\beta \hat{G}_{\alpha\gamma} - \hat{\partial}_\gamma \hat{G}_{\alpha\beta})\hat{G}^{\gamma\sigma}.$$

From the curvature tensor $\hat{R}_{\mu\nu\rho}{}^\sigma$ we get the curvature scalar by contracting the indices

$$\hat{R} := \hat{G}^{\mu\nu}\hat{R}_{\nu\mu\rho}{}^\rho.$$

$\hat{R}$ indeed transforms as a scalar which may be checked explicitly by taking the deformed coproduct (22) into account.

[5] The generalization of covariant derivatives acting on tensors is straight forward [8].

[6] We don't introduce a new symbol for the metric connection.

To obtain an integral which is invariant with respect to the Hopf algebra of deformed infinitesimal diffeomorphisms we need a measure function $\hat{E}$. We demand the transformation property

$$\hat{\delta}_\xi \hat{E} = -\hat{X}_\xi \hat{E} - \hat{X}_{(\partial_\mu \xi^\mu)} \hat{E}. \tag{28}$$

Then it follows with the deformed coproduct (22) that for any scalar field $\hat{S}$

$$\hat{\delta}_\xi \hat{E}\hat{S} = -\hat{\partial}_\mu (\hat{X}_{\xi^\mu}(\hat{E}\hat{S})).$$

Hence, transforming the product of an arbitrary scalar field with a measure function $\hat{E}$ we obtain a total derivative which vanishes under the integral. A suitable measure function with the desired transformation property (28) is for instance given by the determinant of the vierbein $\hat{E}_\mu{}^a$

$$\hat{E} = \det(\hat{E}_\mu{}^a) := \frac{1}{4!} \varepsilon^{\mu_1 \cdots \mu_4} \varepsilon_{a_1 \cdots a_4} \hat{E}_{\mu_1}{}^{a_1} \hat{E}_{\mu_2}{}^{a_2} \hat{E}_{\mu_3}{}^{a_3} \hat{E}_{\mu_4}{}^{a_4}.$$

That $\hat{E}$ transforms correctly can be shown by using that the product of four $\hat{E}_{\mu_i}{}^{a_i}$ transforms as a tensor of fourth rank and some combinatorics.

Now we have all ingredients to write down an Einstein-Hilbert action. Note that having chosen a differential calculus as in (6), the integral is uniquely determined up to a normalization factor by requiring[7] [13]

$$\int \hat{\partial}_\mu \hat{f} = 0$$

for all $\hat{f} \in \hat{\mathcal{A}}$. Then we define the *Einstein-Hilbert action* on $\hat{\mathcal{A}}$ as

$$\hat{S}_{\mathrm{EH}} := \int \det(\hat{E}_\mu{}^a)\hat{R} + \text{complex conj.}.$$

It is by construction invariant with respect to deformed diffeomorphisms meaning that

$$\hat{\delta}_\xi \hat{S}_{\mathrm{EH}} = 0.$$

In this section we have presented the fundamentals of a noncommutative geometry on the algebra $\hat{\mathcal{A}}$ and defined an invariant Einstein-Hilbert action. There is however one important step missing which is subject of the following section: We want to make contact of the noncommutative gravity theory with Einstein's gravity theory. This we achieve by introducing the $\star$-product formalism.

[7] We consider functions that "vanish at infinity".

6 Star Products and Expanded Einstein-Hilbert Action

To express the noncommutative fields in terms of their commutative counterparts we first observe that we can map the whole algebraic construction of the previous sections to the algebra of commutative functions via the vector space isomorphism W introduced in Section 1. By equipping the algebra of commutative functions with a new product denoted by $\star$ be can render W an algebra isomorphism. We define

$$f \star g := W^{-1}(W(f)W(g)) = W^{-1}(\hat{f}\hat{g}) \tag{29}$$

and obtain

$$(\mathcal{A}, \star) \cong \hat{\mathcal{A}}.$$

The $\star$-product corresponding to the symmetric ordering prescription W is then given explicitly by the Moyal-Weyl product[8]

$$f \star g = \mu \circ e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g = fg + \frac{i}{2}\theta^{\mu\nu}(\partial_\mu f)(\partial_\nu g) + \mathcal{O}(\theta^2).$$

It is a deformation of the commutative point-wise product to which it reduces in the limit $\theta \to 0$.

In virtue of the isomorphism W we can map all noncommutative fields to commutative functions in $\mathcal{A}$

$$\hat{F} \mapsto W^{-1}(\hat{F}) \equiv F.$$

We then expand the image F in orders of the deformation parameter θ

$$F = F^{(0)} + F^{(1)} + F^{(2)} + \mathcal{O}(\theta^3),$$

where the zeroth order always corresponds to the undeformed quantity. Products of functions in $\hat{\mathcal{A}}$ are simply mapped to $\star$-products of the corresponding functions in $\mathcal{A}$. The same can be done for the action of the derivative $\hat{\partial}_\mu$ and consequently for an arbitrary differential operator acting on $\hat{\mathcal{A}}$ [8].

The fundamental dynamical field of our gravity theory is the vierbein field $\hat{E}_\mu{}^a$. All other quantities such as metric, connection and curvature can be expressed in terms of it. Its image with respect to W^{-1} is denoted by $E_\mu{}^a$. In first approximation we study the case

$$E_\mu{}^a = e_\mu{}^a,$$

[8] This is an immediate consequence of (10).

where ${e_\mu}^a$ is the usual vierbein field. Then for instance the metric is given up to second order in θ by

$$
\begin{aligned}
G_{\mu\nu} &= \frac{1}{2}({E_\mu}^a \star {E_\nu}^b + {E_\nu}^b \star {E_\mu}^a)\eta_{ab} = \frac{1}{2}({e_\mu}^a \star {e_\nu}^b + {e_\nu}^b \star {e_\mu}^a)\eta_{ab} \\
&= g_{\mu\nu} - \frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}(\partial_{\alpha_1}\partial_{\alpha_2}{e_\mu}^a)(\partial_{\beta_1}\partial_{\beta_2}{e_\nu}^b)\eta_{ab} + \dots,
\end{aligned}
$$

where $g_{\mu\nu}$ is the usual, undeformed metric. For the Christoffel symbol one finds up to second order: The zeroths order is the undeformed expression

$$
\Gamma^{(0)\rho}_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\gamma} + \partial_\nu g_{\mu\gamma} - \partial_\gamma g_{\mu\nu})g^{\gamma\rho}, \tag{30}
$$

the first order reads

$$
\Gamma^{(1)\rho}_{\mu\nu} = \frac{i}{2}\theta^{\alpha\beta}(\partial_\alpha\Gamma^{(0)\sigma}_{\mu\nu})g_{\sigma\tau}(\partial_\beta g^{\tau\rho}) \tag{31}
$$

and the second order

$$
\begin{aligned}
\Gamma^{(2)\rho}_{\mu\nu} = &= -\frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\Big((\partial_{\alpha_1}\partial_{\alpha_2}\Gamma^{(0)}_{\mu\nu\sigma})(\partial_{\beta_1}\partial_{\beta_2}g^{\sigma\rho}) - 2(\partial_{\alpha_1}\Gamma^{(0)}_{\mu\nu\sigma})\partial_{\beta_1}((\partial_{\alpha_2}g^{\sigma\tau})(\partial_{\beta_2}g_{\tau\xi})g^{\xi\rho}) \\
&- \Gamma^{(0)}_{\mu\nu\sigma}\Big((\partial_{\alpha_1}\partial_{\alpha_2}g^{\sigma\tau})(\partial_{\beta_1}\partial_{\beta_2}g_{\tau\xi}) + g^{\sigma\tau}(\partial_{\alpha_1}\partial_{\alpha_2}{e_\tau}^a)(\partial_{\beta_1}\partial_{\beta_2}{e_\xi}^b)\eta_{ab} \\
&- 2\partial_{\alpha_1}((\partial_{\alpha_2}g^{\sigma\tau})(\partial_{\beta_2}g_{\tau\lambda})g^{\lambda\kappa})(\partial_{\beta_1}g_{\kappa\xi})\Big)g^{\xi\rho} + \frac{1}{2}\Big(\partial_\mu((\partial_{\alpha_1}\partial_{\alpha_2}{e_\nu}^a)(\partial_{\beta_1}\partial_{\beta_2}{e_\sigma}^b)) \\
&+ \partial_\nu((\partial_{\alpha_1}\partial_{\alpha_2}{e_\sigma}^a)(\partial_{\beta_1}\partial_{\beta_2}{e_\mu}^b)) - \partial_\sigma((\partial_{\alpha_1}\partial_{\alpha_2}{e_\mu}^a)(\partial_{\beta_1}\partial_{\beta_2}{e_\nu}^b))\Big)\eta_{ab}g^{\sigma\rho}\Big),
\end{aligned} \tag{32}
$$

where

$$
\Gamma^{(0)}_{\mu\nu\sigma} = \Gamma^{(0)\rho}_{\mu\nu}g_{\rho\sigma}. \tag{33}
$$

The expressions for the curvature tensor read

$$
\begin{aligned}
R^{(1)\ \sigma}_{\mu\nu\rho} &= -\frac{i}{2}\theta^{\kappa\lambda}\Big((\partial_\kappa R^{(0)\ \tau}_{\mu\nu\rho})(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} - (\partial_\kappa\Gamma^{(0)\beta}_{\nu\rho})\Big(\Gamma^{(0)\tau}_{\mu\beta}(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} \\
&\quad -\Gamma^{(0)\sigma}_{\mu\tau}(\partial_\lambda g_{\beta\gamma})g^{\gamma\tau} + \partial_\mu((\partial_\lambda g_{\beta\gamma})g^{\gamma\sigma}) + (\partial_\lambda\Gamma^{(0)\sigma}_{\mu\beta})\Big) \\
&\quad +(\partial_\kappa\Gamma^{(0)\beta}_{\mu\rho})\Big(\Gamma^{(0)\tau}_{\nu\beta}(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} - \Gamma^{(0)\sigma}_{\nu\tau}(\partial_\lambda g_{\beta\gamma})g^{\gamma\tau} \\
&\quad +\partial_\nu((\partial_\lambda g_{\beta\gamma})g^{\gamma\sigma}) + (\partial_\lambda\Gamma^{(0)\sigma}_{\nu\beta})\Big)\Big)
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
R^{(2)\ \sigma}_{\mu\nu\rho} &= \partial_\nu\Gamma^{(2)\sigma}_{\mu\rho} + \Gamma^{(2)\gamma}_{\nu\rho}\Gamma^{(0)\sigma}_{\mu\gamma} + \Gamma^{(0)\gamma}_{\nu\rho}\Gamma^{(2)\sigma}_{\mu\gamma} \\
&\quad +\frac{i}{2}\theta^{\alpha\beta}\Big((\partial_\alpha\Gamma^{(1)\gamma}_{\nu\rho})(\partial_\beta\Gamma^{(0)\sigma}_{\mu\gamma}) + (\partial_\alpha\Gamma^{(0)\gamma}_{\nu\rho})(\partial_\beta\Gamma^{(1)\sigma}_{\mu\gamma})\Big) \\
&\quad -\frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}(\partial_{\alpha_1}\partial_{\alpha_2}\Gamma^{(0)\gamma}_{\nu\rho})(\partial_{\beta_1}\partial_{\beta_2}\Gamma^{(0)\sigma}_{\mu\gamma}) \quad - (\mu\leftrightarrow\nu),
\end{aligned} \tag{35}
$$

where the second order is given implicitly in terms of the Christoffel symbol.

The deformed Einstein-Hilbert action is given by

$$\begin{aligned} S_{\mathrm{EH}} &= \frac{1}{2}\int \mathrm{d}^4x \, \det{}_\star e_\mu{}^a \star R + \text{c.c.} \\ &= \frac{1}{2}\int \mathrm{d}^4x \, \det{}_\star e_\mu{}^a \star (R+\bar{R}) \\ &= \frac{1}{2}\int \mathrm{d}^4x \, \det{}_\star e_\mu{}^a (R+\bar{R}) \\ &= S_{\mathrm{EH}}^{(0)} + \int \mathrm{d}^4x \, (\det e_\mu{}^a) R^{(2)} + (\det{}_\star e_\mu{}^a)^{(2)} R^{(0)}, \end{aligned} \tag{36}$$

where we used that the integral together with the Moyal-Weyl product has the property[9]

$$\int \mathrm{d}^4x \, f \star g = \int \mathrm{d}^4x \, fg = \int \mathrm{d}^4x \, g \star f.$$

In (36) $\det_\star e_\mu{}^a$ is the $\star$-determinant

$$\begin{aligned} \det{}_\star e_\mu{}^a &= \frac{1}{4!}\varepsilon^{\mu_1\cdots\mu_4}\varepsilon_{a_1\cdots a_4} e_{\mu_1}{}^{a_1} \star e_{\mu_2}{}^{a_2} \star e_{\mu_3}{}^{a_3} \star e_{\mu_4}{}^{a_4} \\ &= \det e_\mu{}^a + (\det{}_\star e_\mu{}^a)^{(2)} + \dots, \end{aligned}$$

where

$$\begin{aligned} (\det{}_\star)^{(2)} &= -\frac{1}{8}\frac{1}{4!}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\varepsilon^{\mu_1\ldots\mu_4}\varepsilon_{a_1\ldots a_4} \\ &\quad \Big((\partial_{\alpha_1}\partial_{\alpha_2} e_{\mu_1}{}^{a_1})(\partial_{\beta_1}\partial_{\beta_2} e_{\mu_2}{}^{a_2}) e_{\mu_3}{}^{a_3} e_{\mu_4}{}^{a_4} \\ &\quad + \partial_{\alpha_1}\partial_{\alpha_2}(e_{\mu_1}{}^{a_1} e_{\mu_2}{}^{a_2})(\partial_{\beta_1}\partial_{\beta_2} e_{\mu_3}{}^{a_3}) e_{\mu_4}{}^{a_4} \\ &\quad + \partial_{\alpha_1}\partial_{\alpha_2}(e_{\mu_1}{}^{a_1} e_{\mu_2}{}^{a_2} e_{\mu_3}{}^{a_3})(\partial_{\beta_1}\partial_{\beta_2} e_{\mu_4}{}^{a_4}) \Big). \end{aligned} \tag{37}$$

The odd orders of θ vanish in (36) but the even orders of θ give nontrivial contributions.

Equation (36) shows explicitly the corrections to Einsteins gravity predicted by the noncommutative theory.

[9] This follows by partial integration.

Remarks

For an introduction to field theories on noncommutative spaces, we recommend the review articles [13, 14]. To learn more about related approaches to noncommutative geometry the reader is referred to [15, 16]. More about Hopf algebras and Quantum Groups can be found in [10–12]. A good pedagogical introduction to $\star$-products can be found in [17]. The construction of a gravity theory presented in this lecture is based on [8, 9].

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